Feynman rules for Gauss's law

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Abstract

I work on a set of Feynman rules that were derived in order to incorporate the constraint of Gauss's law in the perturbation expansion of gauge field theories and calculate the interaction energy of two static sources. The constraint is implemented via a Lagrange multiplier field, λ, which, in the case of the non-Abelian theory, develops a radiatively generated effective potential term. After analysing the constant and soliton solutions for λ, the confining properties and the various phases of the theory are demonstrated.

1 Introduction

In a previous work I investigated the possibility of expressing the constraint of Gauss's law in the perturbative expansion of gauge field theories [1] via a Lagrange multiplier field, λ, in order to calculate the interaction energy for static sources, and argued for the generation of an effective potential term of the Coleman-Weinberg type for λ and its relation to questions of confinement in the non-Abelian case.

Here, I continue with the calculation of the interaction energy of two static fermionic sources because of the corresponding solutions to the effective action derived. I find a term that is linearly rising with respect to the separation distance in specific scales of the theory that are described and analysed in the text. The result holds in the perturbative regime, together with the usual Coulomb interaction, the only non-perturbative input being the solitons. The fact that the fermionic static charges in the non-Abelian theory are not part of a gauge and Lorentz covariant current creates additional terms whose contributions are estimated and the limits where the various approximations hold are analysed

In Sec. 2, I start with a description of the combinatorics for the Abelian case which do not change the theory but amount to a reshuffling of the propagators, and in Sec. 3 I treat the non-Abelian, self-interacting case. I discuss the limitations and additional corrections and I conclude with a summary and possible extensions in Sec. 4.

2 The Abelian theory

In order to investigate the consequences of the constraint of Gauss's law in the perturbation expansion of gauge field theories I will start with the Abelian case, including a massive fermion, with Lagrangian $\mathcal{L}$ and action

$$S = \int_x \mathcal{L} = \int_x -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \,, \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu + ieA_\mu$. Integrations are over d^4x and the metric conventions are $g_{\mu\nu} = (+---)$, $\partial_\mu = (\partial_0, \partial_i)$, $\partial^\mu = (\partial_0, -\partial_i)$.

Because the Lagrangian is independent of $\dot{A}_0 = \partial_0 A_0$, the respective equation of motion for that field, namely

$$\frac{\delta S}{\delta A_0} = 0, \tag{2}$$

is not a dynamical equation, but, rather, a constraint corresponding to Gauss's law, which will be incorporated in the perturbative expansion via a Lagrange multiplier field, λ, in the path integral

$$Z(J_\mu, \Lambda) = \int [dA_\mu][d\psi][d\bar{\psi}][d\lambda] e^{i\int \tilde{\mathcal{L}}}, \tag{3}$$

where

$$\begin{aligned}
\tilde{\mathcal{L}} &= \frac{1}{2}(\partial_0 A_i + \partial_i A_0)^2 - \frac{1}{4}F_{ij}^2 \\
&+ \bar{\psi}(i\gamma^\mu\partial_\mu - e\gamma^0 A_0 + e\gamma^i A_i - m)\psi \\
&- \lambda(\nabla^2 A_0 + \partial_0\partial_i A_i + e\bar{\psi}\gamma^0\psi) \\
&- \frac{1}{2}(\partial_0 A_0 + \partial_i A_i + \partial_0\lambda)^2 + A_0 J_0 - A_i J_i + \lambda\Lambda.
\end{aligned} \tag{4}$$

In the above equation the first and the second lines contain the original gauge and fermion terms, the third line is the constraint $\lambda\frac{\delta S}{\delta A_0}$, implemented with a gauge-invariant λ, and the last line contains the gauge-fixing term and the sources J_μ, Λ.

I have used a special gauge-fixing condition since the associated term, which can be derived by the usual Faddeev-Popov procedure, gives the simplest set of Feynman rules. The problems of gauge independence and gauge invariance are important and will be discussed later in relation to the non-Abelian theory.

After the usual inversion procedures one obtains the following propagators:

$$G_{00} = -\frac{1}{k^2} - \frac{1}{\vec{k}^2} \tag{5}$$

$$G_{\lambda\lambda} = -\frac{1}{\vec{k}^2} \tag{6}$$

$$G_{0\lambda} = \frac{1}{\vec{k}^2} = G_{\lambda 0} \tag{7}$$

$$G_{ii} = \frac{1}{k^2}. \tag{8}$$

One can easily deduce the vertices from (4) as well as the fact that the propagators are combined in all interactions so as to reproduce all the usual QED diagrams. G_{00}, $G_{\lambda\lambda}$ and $G_{0\lambda}$ appear together and their sum gives the ordinary $0-0$ propagator in Feynman gauge. For example, for two static

current sources separated by a spatial distance, $\vec{r}$, one obtains the usual Coulomb interaction energy from the sum of the diagrams in Fig. 1 in the static limit of $k_0 = 0$,

$$V_{\rm Coul}(r) = 4\pi\alpha_e \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2} = \frac{\alpha_e}{r}, \tag{9}$$

with $\alpha_e = \frac{e^2}{4\pi}$.

3 The non-Abelian theory

I now consider the case of a non-Abelian gauge theory with gauge group G with structure constants f^{abc}, a massive fermion in the representation R with generators T^a, and initial action

$$S_0 = \int_x -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi\,, \tag{10}$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$ and $D_\mu = \partial_\mu + ig\, A^a_\mu\, T^a$.

The theory is gauge invariant, with $\psi \to \omega(\theta)\psi$, $A_\mu \to \omega A_\mu \omega^{-1} + ig^{-1}(\partial_\mu \omega)\omega^{-1}$ under the local gauge transformation $\omega(\theta) = e^{-iT^a\theta^a(x)} \in G$ (with the usual notation $A_\mu = T^a A^a_\mu$).

After imposing the constraint in

$$\tilde{S} = S_0 + \int \lambda^a \frac{\delta S_0}{\delta A_0}, \tag{11}$$

the theory is still gauge-invariant with $\lambda \to \omega\lambda\omega^{-1}$, and can be gauge-fixed as

$$\tilde{S}_{\rm gf} = \tilde{S} + \int_x B^a G^a + \frac{1}{2} B^a B^a + \bar{c}^a \frac{\delta G^a}{\delta \theta^b} c^b, \tag{12}$$

with an auxiliary field, B^a, that can be integrated out, and the gauge-fixing condition $G^a = \partial^\mu A^a_\mu + \partial_0 \lambda^a$, similar to the Abelian case. The resulting action is BRST-invariant when the associated, nilpotent operator of the BRST symmetry, $\mathcal{Q}$, with an infinitesimal, anticommuting parameter ϵ, is also acting on λ with $\mathcal{Q}\lambda^a = -\epsilon \frac{1}{2} f^{abc} \lambda^b c^c$.

The resulting gauge field propagators are the same as the Abelian theory and diagonal in color indices. In particular, one again obtains the static Coulomb interaction from the diagrams of Fig. 1, including color indices,

$$V_{\rm Coul} = -\frac{4\pi C_R \alpha_s}{\vec{k}^2}, \tag{13}$$

in momentum space in the singlet channel, with $\alpha_s = \frac{g^2}{4\pi}$ and C_R the quadratic Casimir of the representation R.

The incorporation of the constraint of Gauss's law via the term $\lambda^a \frac{\delta S_0}{\delta A_0^a}$ has the additional effect of introducing interactions between the gauge field and λ. These are the same as the usual interactions, with one A_0 leg replaced by λ. For example, in Fig. 2, a vertex of the non-Abelian theory is shown together with the new corresponding vertex with the same value.

The usual QCD interactions can be reproduced, with the exception that, for diagrams with external λ legs, the Coulomb interaction is missing in the internal propagators: in Fig. 3, this is shown for the $A_i - A_j$ propagator, with momentum k, and external, constant λ fields, where the missing Coulomb interaction gives a factor of $g^2 C_2 \lambda^2 \frac{k_i k_j}{\vec{k}^2}$, where $\lambda^2 = \lambda^a \lambda^a$ and $f^{acd} f^{bcd} = C_2 \delta^{ab}$.

This amounts to a mass term in loops like Fig. 4, where the $\lambda - A_0$ and $\lambda - \lambda$ interaction cannot be inserted in the loop, and has the effect of generating an effective potential from these terms [1], which would otherwise add up to zero. It is of the Coleman-Weinberg form [2],

$$U(\lambda) = \frac{(\alpha_s C_2)^2}{4} \lambda^4 \left(\ln \frac{\lambda^2}{\mu^2} - \frac{1}{2} \right), \tag{14}$$

renormalized at a scale μ where $dU/d\lambda = 0$, and appears in the effective action with the opposite sign (it is upside-down).

The auxiliary field λ has similar interactions with A_0, which, of course, does not develop an effective potential term because of gauge and Lorentz invariance. This shows in the fact that the corresponding integral expressions are not covariant and can be, with a suitable regularization, set to zero [3]. The same is true for the ghost terms that appear from the gauge-fixing condition, the terms coming from the four-vertices, and the gauge-dependent expressions for a general gauge-fixing. These will not contribute to the effective potential derived for λ, but to wave-function renormalisation factors.

Terms of the form $(D_i \lambda^a)^2$ are also generated in the effective action, since they are gauge and BRST-invariant. However, these, like the original terms of the effective action, are multiplied by wave-function renormalization terms that are of higher order in the perturbative expansion. A general such wave-function renormalization term can be written in the form $Z(\lambda) = Z_{\text{tree}} + const \cdot \alpha_s^2 \ln(\lambda^2/\mu^2)$, with calculable constants and suitable renormalization conditions so that, at $\lambda = \mu$, it retains its tree-level value [2]. I will assume an approximation that neglects the higher-order terms in these expansions.

These terms of the form $(D_i\lambda^a)^2$, contain expressions that are not Lorentz covariant, and are generated in the effective action, in the non-Abelian case, since the charges that appear in Gauss's law (with $E_i^a = F_{0i}^a$)

$$\frac{\delta S_0}{\delta A_0^a} = D_i E_i^a - g\rho^a = 0, \tag{15}$$

are part of a gauge but not Lorentz covariantly conserved current, $\rho^a = j_0^a$, with $j_\mu^a = \bar\psi\gamma_\mu T^a\psi$, that satisfies $D^\mu j_\mu = 0$, with the gauge covariant derivative in the adjoint representation, but not $\partial^\mu j_\mu = 0$. The calculation of the static interaction energy from $E(r) =< \Omega|T(\rho^a(r)\rho^a(0))|\Omega >$, is well-defined, however, since these terms appear only at higher order in the effective action. In fact, this is how the usual Coulomb interaction is calculated in QCD as well as in the present work, namely from the diagrams in Fig. 1. The presence of these terms at higher order does not invalidate the analysis given here.

For values of $\lambda \approx \mu$, and for slowly-varying λ, therefore, I expect this approximation to hold, and will accordingly examine the following effective action around $\lambda \approx \mu$, where $Z(\mu) \approx Z_{\text{tree}}$ for the various wave-function renormalization factors.

$$S_{\text{eff}} = \int_x -\frac{1}{4}F_{\mu\nu}^2 + \bar\psi(i\gamma^\mu D_\mu - m)\psi + \lambda^a\frac{\delta S_0}{\delta A_0^a} - \frac{1}{2}G^2 + U(\lambda). \tag{16}$$

Considering the stationary points of this action with respect to A_0^a and λ^a leads to the basic equation for λ^a:

$$\nabla^2\lambda^a = \frac{\partial U}{\partial\lambda^a}, \tag{17}$$

that admits "soliton-like" solutions $\lambda_s^a(r)$, which are three-dimensional, spherically symmetric bubbles with

$$\lambda_s^a \sim \mu \ \text{ in a radius } \ R_s \sim \frac{1}{\alpha_s\, C_2\, \mu}. \tag{18}$$

In momentum space, they are also bubbles $\tilde\lambda_s^a(q)$ with

$$\tilde\lambda_s^a \sim \frac{1}{\alpha_s^3\, C_2^3\, \mu^2} \ \text{ in a radius } \ \tilde R_s \sim \frac{1}{R_s} \sim \alpha_s\, C_2\, \mu \tag{19}$$

(I denote by r, q, k the modulus of the corresponding three-dimensional vectors, and by $\sim$ equality modulo numerical factors of order unity).

One may now examine the contribution of these configurations in the interactions of the theory. We still assume that we are in the perturbative regime, with $\alpha_s C_2$ less than unity; the aforementioned solitons, however, are non-perturbative solutions, to be considered as insertions of external fields.

The first contribution to the static interaction energy between two fermionic current sources due to external insertions of λ is given by the diagram of Fig. 5 in the static limit of $k_0 = 0$:

$$\begin{aligned} E(r) = \; - \; & (4\pi)^2 \alpha_s^2 C_R C_2 \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{d^3\vec{q'}}{(2\pi)^3} \frac{d^3\vec{k}}{(2\pi)^3} e^{i(\vec{q}+\vec{q'}+\vec{k})\cdot\vec{r}} \\ \cdot \; & \frac{\tilde{\lambda}_s^a(q)\tilde{\lambda}_s^a(q')\,(\vec{k}-\vec{q})\cdot(\vec{k}+\vec{q}+2\vec{q'})}{\vec{k}^2\,(\vec{k}+\vec{q})^2\,(\vec{k}+\vec{q}+\vec{q'})^2}. \end{aligned} \tag{20}$$

Investigation of the consequences of this expression leads to a description of the expected confining properties of the theory and its various phases [4]:

(A) There is an interesting limit of (20) for the one-soliton configuration, for $k \gtrsim \mu$, when k is well outside the bubble, and one may simplify by setting $\vec{q}, \vec{q'} = 0$ in most terms, getting an energy term

$$E(r) = 4\pi\,\alpha_s^2\, C_R\, C_2\, \lambda_s^2(r)\, r. \tag{21}$$

The bubbles are not of the thin wall type, however, deep inside the bubble, near the spatial origin, essentiallly because $\nabla^2 = \partial_r^2 + \frac{2}{r}\partial_r$, the solutions of (17) are slowly varying and of order μ [5], corresponding to a confining interaction

$$V_{\rm conf} = \sigma\, r \tag{22}$$

with a string tension

$$\sigma \sim \alpha_s^2\, C_R\, C_2\, \mu^2 \tag{23}$$

according to the power-counting results mentioned before.

The Coulomb interaction is always there, since it corresponds to no insertions of external fields, so that

$$E(r) = V_{\rm Coul} + V_{\rm conf}, \tag{24}$$

at this order.

(B) The result (23), as well as the general expression (20), are proportional to the quadratic Casimir, C_R, of the representation of the sources,

therefore they obviously satisfy the requirement of Casimir scaling for the confining force.

(C) For the gauge group $G = SU(N)$, with $C_2 = N$ and $C_F = (N^2 - 1)/2N$ for the adjoint and fundamental representations respectively, one may consider the large-N limit of (23), and write the result for the string tension in terms of $\lambda_{tH} = g^2 N$,

$$\sigma \sim \lambda_{tH}^2 \frac{C_R}{C_F} \mu^2, \tag{25}$$

which, apart from the factor of Casimir scaling, has the expected λ_{tH}^2 dependence seen in lattice investigations.

(D) For larger distances, r, one may consider the $k \approx 0$ limit of (20), and calculate the $1/r$ terms that arise when one is near the origin in momentum space and can set $k \approx 0$ in most terms; after a change of variables and repeated application of the convolution theorem for three-dimensional Fourier transforms, one finds, besides exponentially small terms,

$$E_L(r) = -(4\pi)^2 \alpha_s^2 \, C_R C_2 \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2} \int d^3\vec{R} \frac{1}{R} \lambda_s^2(\vec{r} - \vec{R}). \tag{26}$$

For r large but of order R_s, the second integral is of order $\mu^2 R_s^2$, leading to a “Lüscher-type” term,

$$E_L(r) \sim -\frac{C_R}{C_2} \frac{1}{r}. \tag{27}$$

Apart from a Casimir scaling factor, which disappears for sources in the adjoint representation, the overall coefficient is negative and a pure number, independent of couplings and scales.

For r much larger than R_s, the contribution of the one-soliton configuration falls of as

$$E_1(r) \sim -\frac{C_R}{\alpha_s C_2^2 \mu} \frac{1}{r^2}, \tag{28}$$

however, in this case, one expects the multi-soliton configurations to become dominant, as explained below.

(E) The basic equation for λ, (17), has, besides the one-soliton solution, the obvious solutions $\lambda = 0$ and $\lambda = \mu$, as well as multi-soliton solutions, and their Euclidean action is just $\int_x -U(\lambda)$ if one turns off the gauge fields.

When $\lambda = 0$, (20) obviously vanishes, and the theory is in the Coulomb phase, with only the Coulomb interaction present.

When $\lambda = \mu$, (20) gives a pure confining interaction in the entire space, and (23) is exact for the string tension; the theory is in the string phase.

However, the Euclidean action for $\lambda = \mu$ is positive (and infinite for infinite volume) and for $\lambda = 0$ is zero. For the soliton configuration the Euclidean action is negative, as one can see by the analogy with the three-dimensional tunneling problem [5]. Accordingly, the configuration that minimizes the Euclidean action is one with closely-packed solitons, and the theory is in the confining phase. Starting with the theory in the string phase, with $\lambda = \mu$ everywhere, string-breaking is favorable, leading to the confining configuration of closely-packed solitons.

(F) The high-temperature phase transition can be studied in the usual manner, by the finite-temperature contributions to the effective potential, and, similarly to the phase transition for Coleman-Weinberg models, it is expected to be of first-order when the scale μ is considered as an order parameter. The high-temperature vacuum admits only the $\lambda = 0$ solution, that is the deconfined, Coulomb phase.

4 Comments

In the present work I examined some of the consequences of a set of Feynman rules that were initially mentioned in [1] as an expression of the constraint of Gauss's law, and the associated effective action and soliton solutions, in order to calculate the static interaction energy. There are some similarities and common features with other approaches [6] and effective actions proposed. However, the treatment here is initially perturbative, having the non-perturbative input of the afore-mentioned solitons. The action considered here is also BRST-invariant, leading to the expectation that the entire procedure is renormalizable.

I used a Lagrange multiplier, auxiliary field, λ, in order to impose the constraint of Gauss's law. It is usually stated, or derived in standard quantization procedures, that A_0 is the Lagrange multiplier associated with this constraint. However, it is not a direct consequence of the path integral quantization with constraints that λ becomes A_0, as the Lagrange multiplier that enforces the constraint that A_0 itself satisfies. Doing so maintains the manifest Lorentz invariance, and in the Abelian case the procedures are equivalent, but in the non-Abelian case the present method gives some additional terms, specifically for the calculation of the interaction energy for static sources.

The combinatorics described in [1] and here can be interpreted as enforcing the Coulomb interaction to behave "classically", that is, to propagate only in tree diagrams, not in loops. This is not an *ad hoc* assumption, but a consequence of enforcing Gauss's law at all orders in perturbation theory, not just the "final" or "physical" states. The fact that the Coulomb interaction is missing from loops has the effect of generating the effective potential term for λ and the consequences described above.

The theory is still Lorentz invariant at the fundamental level, one has, however, considered static fermionic sources in order to calculate the static interaction energy. In the non-Abelian case, since these sources are part of a gauge but not Lorentz covariantly conserved current, there are additional terms that are generated at higher order in the effective action that are not Lorentz covariant. These do not spoil the calculation of the interaction energy at this order, but may contribute at higher orders. First of all, these terms are calculable since the theory is renormalizable, and their contributions to the final result can be seen to be of higher order (at least of order α_s^4). Also, the basic soliton equation, (17), is not affected by these corrections, except for possible multiplicative factors close to unity, and the soliton solutions still exist. That is, the extra terms do not change the basic result of the dynamics, namely the Coulomb and the confining nature of the interaction.

The characteristic property of asymptotic freedom is not directly used in the treatment of the non-Abelian theory, although it is important for consistency in the scales involved, for example, for the fact that perturbation theory is valid deep inside the spatial bubbles. Asymptotic freedom will show up, also, in a fuller treatment that includes the wavefunction renormalization terms, $Z(\lambda)$, and their dielectric properties.

The generation of an effective potential of the Coleman-Weinberg type, along with non-perturbative solutions, that are not scale invariant like the QCD instantons, makes the present approach particularly convenient for the investigation of the question of confinement and, generally, the issue of the phases of gauge theories. The various solutions of (17) give the different phases involved, as described before, and one may examine the effective action at finite temperature along with the corresponding deconfinement phase transition. The fact, for example, that the high-temperature phase transition in Coleman-Weinberg models is of the first-order yields a similar expectation for the deconfinement phase transition when investigated by the present method. It is also possible to do similar calculations in the case of a spontaneously broken gauge theory and consider the relations between the

Higgs, the confined and the Coulomb phases.

These problems and other extensions will be deferred to future work.

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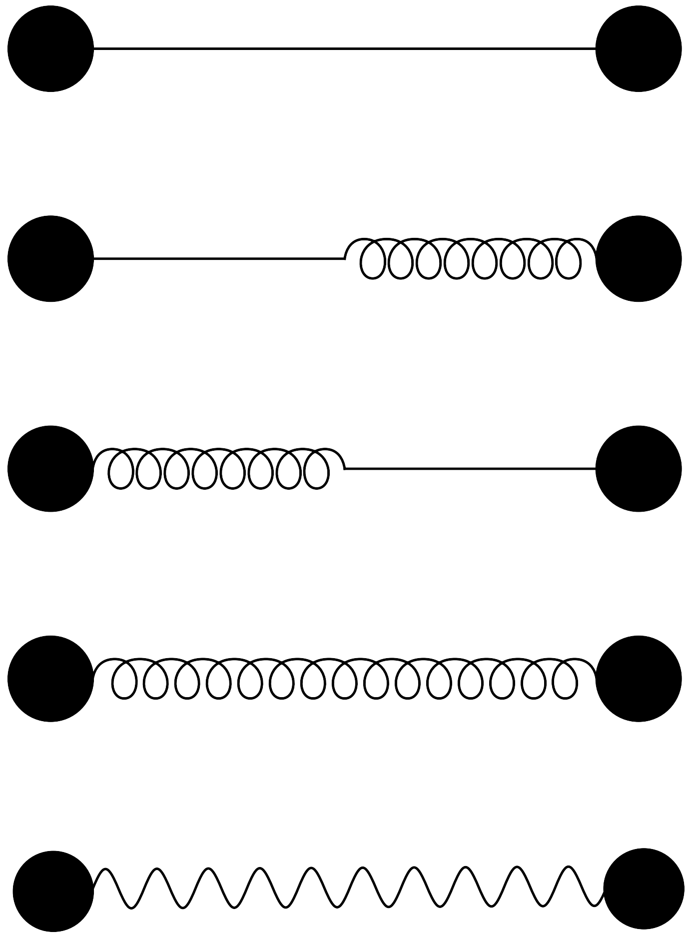

Figure 1: The propagators combine to reproduce the Coulomb interaction between two static sources (large blobs). Solid, wavy and curly lines denote the A_0, A_i and λ fields respectively.

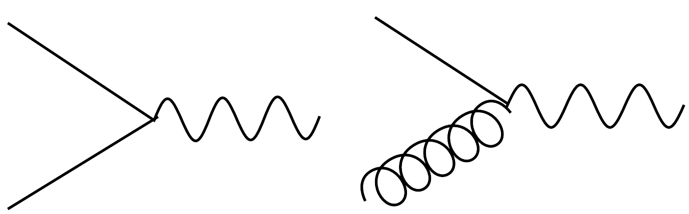

Figure 2: Two vertices for the non-Abelian theory.

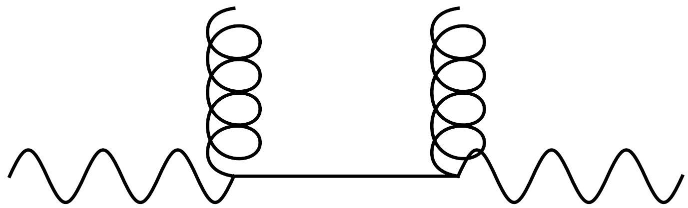

Figure 3: The modifications to the i-j propagator.

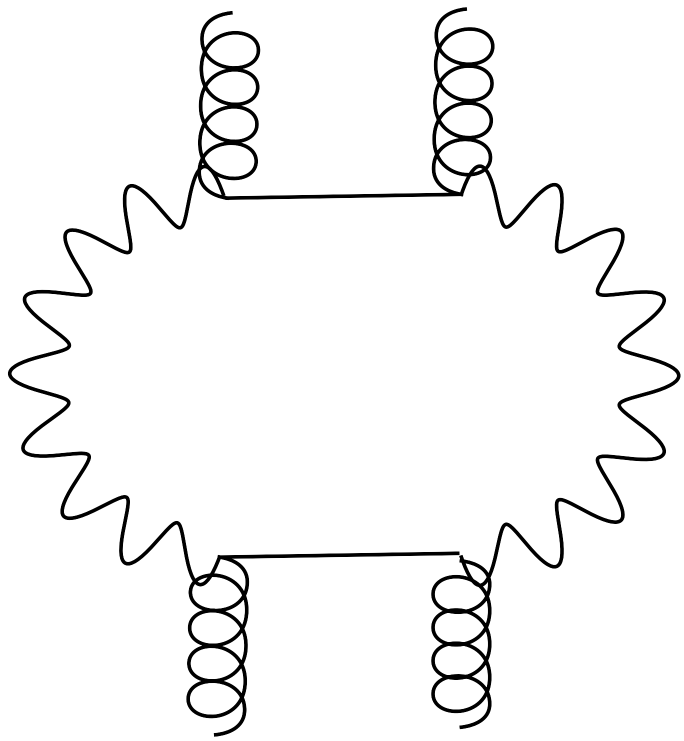

Figure 4: A graph for the generated effective potential.

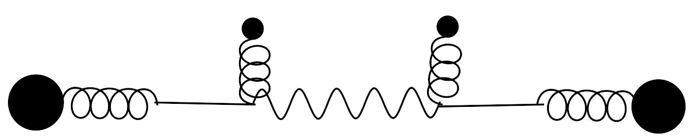

Figure 5: The graph for the interaction energy of two static sources (large blobs) with two insertions of λ (smaller blobs).